India National Olympiad 2000

www.artofproblemsolving.com/community/c4926

by Mathx, grobber, Rushil, Together

1 The incircle of ABC touches BC, CA, AB at K, L, M respectively. The line through A parallel to LK meets MK at P, and the line through A parallel to MK meets LK at Q. Show that the line PQ bisects AB and bisects AC.

---

2 Solve for integers x, y, z:

$$\begin{cases} x+y &= 1-z \\ x^3+y^3 &= 1-z^2. \end{cases}$$

---

3 If a, b, c, x are real numbers such that $abc \neq 0$ and

$$\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c},$$

then prove that $a = b = c$.

---

4 In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SQP = 30^\circ$. Prove that $\angle PQR - \angle QRS = 90^\circ$.

---

5 Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex), then $|\lambda| \leq 1$.

---

6 For any natural numbers n, ($n \geq 3$), let $f(n)$ denote the number of congruent integer-sided triangles with perimeter n. Show that

(i) $f(1999) > f(1996)$;

(ii) $f(2000) = f(1997)$.

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India National Olympiad 2001

www.artofproblemsolving.com/community/c4927

by Rushil

1 Let ABC be a triangle in which no angle is 90°. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove that

(i) If P is the incenter or an excentre of ABC, then P is the circumenter of $A_1B_1C_1$;

(ii) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

(iii) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

---

2 Show that the equation $x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$ has infintely many solutions in integers x, y, z.

---

3 If a, b, c are positive real numbers such that $abc = 1$, Prove that

$$a^{b+c}b^{c+a}c^{a+b} \leq 1.$$

---

4 Show that given any nine integers, we can find four, a, b, c, d such that $a + b - c - d$is divisible by 20. Show that this is not always true for eight integers.

---

5 ABC is a triangle. M is the midpoint of BC. $\angle MAB = \angle C$, and $\angle MAC = 15^\circ$. Show that $\angle AMC$ is obtuse. If O is the circumcenter of ADC, show that AOD is equilateral.

---

6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)f(xy)$ for all $x, y \in \mathbb{R}$.

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India National Olympiad 2002

www.artofproblemsolving.com/community/c4928

by Rushil, Fermat -Euler, seshadri

1 For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following statements.

(a_1) AB is parallel to DE. (a_2)$AE = BD$.

(b_1) BC is parallel to EF. (b_2)$BF = CE$.

(c_1) CD is parallel to FA. (c_2) $CA = DF$.

(a) Show that if all six of these statements are true then the hexagon is cyclic.

(b) Prove that, in fact, five of the six statements suffice.

---

2 Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c . Find all a, b, c which give the smallest value

---

3 If x, y are positive reals such that $x + y = 2$ show that $x^3y^3(x^3 + y^3) \leq 2$.

---

4 Is it true that there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

---

5 Do there exist distinct positive integers a, b, c such that a, b, c, $-a + b + c$, $a - b + c$, $a + b - c$, $a + b + c$ form an arithmetic progression (in some order).

---

6 The numbers $1, 2, 3, \ldots, n^2$ are arranged in an $n \times n$ array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible values for a_{jj}. Show that

$$b_1 + b_2 + \cdots + b_n = \frac{n(n^2 - 3n + 5)}{3}.$$

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India National Olympiad 2003

www.artofproblemsolving.com/community/c4929

by Fermat -Euler, hardsoul

1 Let P be an interior point of an acute-angled triangle ABC. The line BP meets the line AC at E, and the line CP meets the line AB at F. The lines AP and EF intersect each other at D. Let K be the foot of the perpendicular from the point D to the line BC. Show that the line KD bisects the angle $\angle EKF$.

2 Find all primes p, q and even $n > 2$ such that $p^n + p^{n-1} + ... + 1 = q^2 + q + 1$.

3 Show that $8x^4 - 16x^3 + 16x^2 - 8x + k = 0$ has at least one real root for all real k. Find the sum of the non-real roots.

4 Find all 7-digit numbers which use only the digits 5 and 7 and are divisible by 35.

5 Let a, b, c be the sidelengths and S the area of a triangle ABC. Denote $x = a + \frac{b}{2}$, $y = b + \frac{c}{2}$ and $z = c + \frac{a}{2}$. Prove that there exists a triangle with sidelengths x, y, z, and the area of this triangle is $\geq \frac{9}{4}S$.

6 Each lottery ticket has a 9-digit numbers, which uses only the digits $1, 2, 3$. Each ticket is colored red, blue or green. If two tickets have numbers which differ in all nine places, then the tickets have different colors. Ticket 122222222 is red, and ticket 222222222 is green. What color is ticket 123123123 ?

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India National Olympiad 2004

www.artofproblemsolving.com/community/c4930

by Fermat -Euler

1	$ABCD$ is a convex quadrilateral. K, L, M, N are the midpoints of the sides AB, BC, CD, DA. BD bisects KM at Q. $QA = QB = QC = QD$, and $\frac{LK}{LM} = \frac{CD}{CB}$. Prove that $ABCD$ is a square
2	$p > 3$ is a prime. Find all integers a, b, such that $a^2 + 3ab + 2p(a + b) + p^2 = 0$.
3	If a is a real root of $x^5 - x^3 + x - 2 = 0$, show that $[a^6] = 3$
4	ABC is a triangle, with sides a, b, c , circumradius R, and exradii r_a, r_b, r_cIf $2R \leq r_a$, show that $a > b$, $a > c$, $2R > r_b$, and $2R > r_c$.
5	S is the set of all (a, b, c, d, e, f) where a, b, c, d, e, f are integers such that $a^2+b^2+c^2+d^2+e^2 = f^2$. Find the largest k which divides abcdef for all members of S.
6	Show that the number of 5-tuples (a, b, c, d, e) such that $abcde = 5(bcde+acde+abde+abce+abcd)$ is odd

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India National Olympiad 2005

www.artofproblemsolving.com/community/c4931

by Rushil, shobber, silouan, Skyward_Sea, Diarmuid

1 Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

---

2 Let α and β be positive integers such that $\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$. Find the minimum possible value of β.

---

3 Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$\begin{aligned} px^2 + 2qx + r &= 0 \\ qx^2 + 2rx + p &= 0 \\ rx^2 + 2px + q &= 0. \end{aligned}$$

have a common root, say α. Prove that

$a)$ α is real and negative;

$b)$ the remaining third quadratic equation has non-real roots.

---

4 All possible 6-digit numbers, in each of which the digits occur in nonincreasing order (from left to right, e.g. 877550) are written as a sequence in increasing order. Find the 2005-th number in this sequence.

---

5 Let x_1 be a given positive integer. A sequence $\{x_n\}_{n\geq 1}$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

a) the sequence contains an even term;

b) the sequence contains infinitely many even terms.

---

6 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

for all $x, y, z \in \mathbb{R}$.

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India National Olympiad 2006

www.artofproblemsolving.com/community/c4932

by Vihang, Farenhajt, campos, pavel kozlov

– February 5th

1 In a non equilateral triangle ABC the sides a, b, c form an arithmetic progression. Let I be the incentre and O the circumcentre of the triangle ABC. Prove that

(1) IO is perpendicular to BI;

(2) If BI meets AC in K, and D, E are the midpoints of BC, BA respectively then I is the circumcentre of triangle DKE.

2 Prove that for every positive integer n there exists a unique ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2)+a.$$

3 Let $X = \mathbb{Z}^3$ denote the set of all triples (a, b, c) of integers. Define $f : X \to X$ by

$$f(a, b, c) = (a+b+c, ab+bc+ca, abc).$$

Find all triples (a, b, c) such that

$$f(f(a, b, c)) = (a, b, c).$$

4 Some 46 squares are randomly chosen from a 9×9 chess board and colored in red. Show that there exists a 2×2 block of 4 squares of which at least three are colored in red.

5 In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$ and $\angle ABD = 30^\circ$. Prove that

(1) $c \geq a + b$;

(2) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

6 (a) Prove that if n is a integer such that $n \geq 4011^2$ then there exists an integer l such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$

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(b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$ then there exists an integer l such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$

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India National Olympiad 2007

www.artofproblemsolving.com/community/c4933

by Rijul saini, Sathej

1 In a triangle ABC right-angled at C , the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3$$

---

2 Let n be a natural number such that $n = a^2 + b^2 + c^2$ for some natural numbers a, b, c. Prove that

$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2$$

where p_j's , q_j's , r_j's are all **nonzero** integers. Further, if 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2+y^2+z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

---

3 Let m and n be positive integers such that $x^2 - mx + n = 0$ has real roots α and β.

Prove that α and β are integers **if and only if** $[m\alpha] + [m\beta]$ is the square of an integer.

(Here $[x]$ denotes the largest integer not exceeding x)

---

4 Let $\sigma = (a_1, a_2, \cdots, a_n)$ be permutation of $(1, 2, \cdots, n)$. A pair (a_i, a_j) is said to correspond to an **inversion** of σ if $i < j$ but $a_i > a_j$. How many permutations of $(1, 2, \cdots, n)$, $n \geq 3$, have exactly **two** inversions?

For example, In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1), (4, 3), (4, 1), (5, 3), (5, 1), (3, 1)$.

---

5 Let ABC be a triangle in which $AB = AC$. Let D be the midpoint of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. Define

$$\frac{AP}{PD} = \frac{BP}{PE} = \lambda, \quad \frac{BD}{AD} = m, \quad z = m^2(1 + \lambda)$$

Prove that

$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

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6 If x, y, z are positive real numbers, prove that

$$(x+y+z)^2 (yz+zx+xy)^2 \leq 3\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right).$$

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India National Olympiad 2008

www.artofproblemsolving.com/community/c4934

by Chronoz, manjil, Rijul saini, makar

1 Let ABC be triangle, I its in-center; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concylic, where I_1 is the in-center of triangle A_1, B_1, C_1.

2 Find all triples (p, x, y) such that $p^x = y^4+4$, where p is a prime and x and y are natural numbers.

3 Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

4 All the points with integer coordinates in the xy-Plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is red and the point $(0, 1)$ is blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a right-angled triangle.

5 Let ABC be a triangle; $\Gamma_A, \Gamma_B, \Gamma_C$ be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles $\Gamma_A, \Gamma_B, \Gamma_C$ externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

6 Let $P(x)$ be a polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that
(i) $P(x) \cdot Q(x)$ is a polynomial in x^2 , and
(ii) $P(x) \cdot R(x)$ is a polynomial in x^3.

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India National Olympiad 2009

www.artofproblemsolving.com/community/c4935

by skand

Day 1	
1	Let ABC be a tringle and let P be an interior point such that $\angle BPC = 90, \angle BAP = \angle BCP$.Let M, N be the mid points of AC, BC respectively.Suppose $BP = 2PM$.Prove that A, P, N are collinear.
2	Define a a sequence $< a_n >_{n=1}^{\infty}$ as follows $a_n = 0$, if number of positive divisors of n is *odd* $a_n = 1$, if number of positive divisors of n is *even* (The positive divisors of n include 1 as well as n.)Let $x = 0.a_1a_2a_3.....$ be the real number whose decimal expansion contains a_n in the n-th place,$n \geq 1$.Determine,with proof,whether x is rational or irrational.
3	Find all real numbers x such that: $[x^2 + 2x] = [x]^2 + 2[x]$ (Here $[x]$ denotes the largest integer not exceeding x.)
Day 2	
4	All the points in the plane are colored using three colors.Prove that there exists a triangle with vertices having the same color such that *either* it is isosceles *or* its angles are in geometric progression.
5	Let ABC be an acute angled triangle and let H be its ortho centre. Let h_{max} denote the largest altitude of the triangle ABC. Prove that: $AH + BH + CH \leq 2h_{max}$
6	Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$.Prove that: $a^2 + b^2 - c^2 > 6(c - a)(c - b)$.

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India National Olympiad 2010

www.artofproblemsolving.com/community/c4936

by keyree10

1 Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

---

2 Find all natural numbers $n > 1$ such that n^2 does not divide $(n-2)!$.

---

3 Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = xyz$$

$$(x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) = x^3y^3z^3$$

---

4 How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$ (where a_7 is to be taken as a_1) are all equal to one another?

---

5 Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

---

6 Define a sequence $< a_n >_{n \geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.
(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.
(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

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India National Olympiad 2011

www.artofproblemsolving.com/community/c4937

by Potla

1 Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Show that $\triangle ABC$ is equilateral.

---

2 Call a natural number n faithful if there exist natural numbers $a < b < c$ such that $a|b$, and $b|c$ and $n = a+b+c$. (i) Show that all but a finite number of natural numbers are faithful. (ii) Find the sum of all natural numbers which are not faithful.

---

3 Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients such that $a_n - b_n$ is a prime and $a_n b_0 - a_0 b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.

---

4 Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

---

5 Let $ABCD$ be a cyclic quadrilateral inscribed in a circle Γ. Let E, F, G, H be the midpoints of arcs AB, BC, CD, AD of Γ, respectively. Suppose that $AC \cdot BD = EG \cdot FH$. Show that AC, BD, EG, FH are all concurrent.

---

6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2 f(y),$$

For all $x, y \in \mathbb{R}$.

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India National Olympiad 2012

www.artofproblemsolving.com/community/c4938

by Sahil, Learner94, adihaya, Potla

–	February 5th
1	Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2+\sqrt{2}}$ and AB subtends 135 degrees at center of circle . Find the maximum possible area of $ABCD$.
2	Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers, such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.
3	Define a sequence $< f_0(x), f_1(x), f_2(x), \cdots >$ of functions by $$f_0(x) = 1$$ $$f_1(x) = x$$ $$(f_n(x))^2 - 1 = f_{n+1}(x) f_{n-1}(x)$$ for $n \geq 1$. Prove that each $f_n(x)$ is a polynomial with integer coefficients.
4	Let ABC be a triangle. An interior point P of ABC is said to be *good* if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 *smaller triangles of equal area.* Determine the number of good points for a given triangle ABC.
5	Let ABC be an acute angled triangle. Let D, E, F be points on BC, CA, AB such that AD is the median, BE is the internal bisector and CF is the altitude. Suppose that $\angle FDE = \angle C, \angle DEF = \angle A$ and $\angle EFD = \angle B$. Show that ABC is equilateral.
6	Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and $(i) f(xy) + f(x)f(y) = f(x) + f(y)$ $(ii)\ (f(x-y) - f(0))\, f(x)f(y) = 0$ for all $x, y \in \mathbb{Z}$, simultaneously. (a) Find the set of all possible values of the function f. (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

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India National Olympiad 2013

www.artofproblemsolving.com/community/c4939

by Potla, Learner94

–	February 3rd
1	Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let O_1 and O_2 be the centres of Γ_1 and Γ_2, respectively. Let ℓ_1 be a line which is tangent to Γ_2 at P and passing through O_1, and let ℓ_2 be the line tangent to Γ_1 at Q and passing through O_2. Let $K = \ell_1 \cap \ell_2$. If $KP = KQ$ then prove that the triangle PQR is equilateral.
2	Find all $m, n \in \mathbb{N}$ and primes $p \geq 5$ satisfying $$m(4m^2 + m + 12) = 3(p^n - 1).$$
3	Let $a, b, c, d \in \mathbb{N}$ such that $a \geq b \geq c \geq d$. Show that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.
4	Let N be an integer greater than 1 and let T_n be the number of non empty subsets S of $\{1, 2, ..., n\}$ with the property that the average of the elements of S is an integer.Prove that $T_n - n$ is always even.
5	In an acute triangle ABC, let O, G, H be its circumcentre, centroid and orthocenter. Let $D \in BC, E \in CA$ and $OD \perp BC, HE \perp CA$. Let F be the midpoint of AB. If the triangles ODC, HEA, GFB have the same area, find all the possible values of $\angle C$.
6	Let a, b, c, x, y, z be six positive real numbers satisfying $x + y + z = a + b + c$ and $xyz = abc$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.

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India National Olympiad 2014

www.artofproblemsolving.com/community/c4940

by shivangjindal, backfromthedead

1 In a triangle ABC, let D be the point on the segment BC such that $AB + BD = AC + CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

---

2 Let n be a natural number. Prove that,

$$\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor + \lfloor \sqrt{n} \rfloor$$

is even.

---

3 Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisble by $a + b$. Prove that the quotient is at most $\frac{a+b}{4}$. When is this quotient exactly equal to $\frac{a+b}{4}$

---

4 Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternately (starting with Calvin) in the following game. At his turn, Calvin should either increase or decrease the coefficient of x by 1. And at this turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning stratergy.

---

5 In a acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

---

6 Let $n > 1$ be a natural number. Let $U = \{1, 2, ..., n\}$, and define $A\Delta B$ to be the set of all those elements of U which belong to exactly one of A and B. Show that $|\mathcal{F}| \leq 2^{n-1}$, where $\mathcal{F}$ is a collection of subsets of U such that for any two distinct elements of A, B of $\mathcal{F}$ we have $|A\Delta B| \geq 2$. Also find all such collections $\mathcal{F}$ for which the maximum is attained.

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India National Olympiad 2015

www.artofproblemsolving.com/community/c4941

by TripteshBiswas

–	February 1st
1	Let ABC be a right-angled triangle with $\angle B = 90^\circ$. Let BD is the altitude from B on AC. Let P, Q and Ibe the incenters of triangles ABD, CBD and ABC respectively.Show that circum-center of triangle PIQ lie on the hypotenuse AC.
2	For any natural number $n > 1$ write the finite decimal expansion of $\frac{1}{n}$ (for example we write $\frac{1}{2} = 0.4\overline{9}$ as its infinite decimal expansion not 0.5). Determine the length of non-periodic part of the (infinite) decimal expansion of $\frac{1}{n}$.
3	Find all real functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + yf(x)) = xf(x + y)$.
4	There are four basketball players A, B, C, D. Initially the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after **seven** moves? (for example $A \to C \to B \to D \to A \to B \to C \to A$, or $A \to D \to A \to D \to C \to A \to B \to A$).
5	Let $ABCD$ be a convex quadrilateral.Let diagonals AC and BD intersect at P. Let PE, PF, PG and PH are altitudes from P on the side AB, BC, CD and DA respectively. Show that $ABCD$ has a incircle if and only if $\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}$.
6	Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

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India National Olympiad 2016

www.artofproblemsolving.com/community/c211903

by eshan, YESMAths

– Problems

P1 Let ABC be a triangle in which $AB = AC$. Suppose the orthocentre of the triangle lies on the incircle. Find the ratio $\frac{AB}{BC}$.

P2 For positive real numbers a, b, c which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

P3 Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

P4 Suppose 2016 points of the circumference of a circle are colored red and the remaining points are colored blue . Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

P5 Let ABC be a right-angle triangle with $\angle B = 90^\circ$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

P6 Consider a nonconstant arithmetic progression $a_1, a_2, \cdots, a_n, \cdots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

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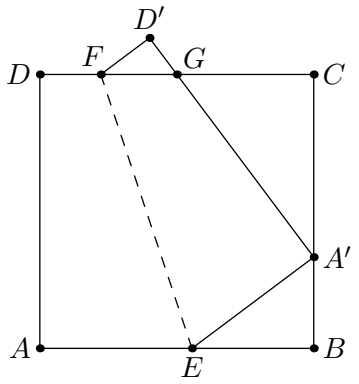
India National Olympiad 2017

www.artofproblemsolving.com/community/c393313

by PSY-Math, BartSimpsons

1 In the given figure, $ABCD$ is a square sheet of paper. It is folded along EF such that A goes to a point A' different from B and C, on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ and $A'BE$.



---

2 Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

---

3 Find the number of triples (x, a, b) where x is a real number and a, b belong to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that

$$x^2 - a\{x\} + b = 0.$$

where $\{x\}$ denotes the fractional part of the real number x.

---

4 Let $ABCDE$ be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120^\circ$ and the side lengths are five *consecutive integers* in some order. Find all possible values of $AB + BC + CD$.

---

5 Let ABC be a triangle with $\angle A = 90^\circ$ and $AB < AC$. Let AD be the altitude from A on to BC, Let P, Q and I denote respectively the incentres of triangle ABD, ACD and ABC. Prove that AI is perpendicular to PQ and $AI = PQ$.

---

6 Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-k} \cdot 3^2 + \cdots.$$

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Show that $2x-1, 2x, 2x+1$ form the sides of a triangle whose area and inradius are also integers.

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India National Olympiad 2018

www.artofproblemsolving.com/community/c596374

by integrated_JRC

1 Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points of BC and CA ; let G be the centroid of ΔABC. Suppose, D, C, E, G are concyclic. Find the least possible perimeter of ΔABC.

---

2 For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by 3 types of tiles : 1×1 red tile, 1×1 green tile and 1×2 domino. (For example, we can have 5 types of tiling when $n = 2$: red-red ; red-green ; green-red ; green-green ; and blue.) Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these 3 types of tiles. Prove that, t_n divides t_{2n+1}.

---

3 Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of ΔO_1AO_2 intersect Γ_1 and Γ_2 respectively in points $C(\neq A)$ and $D(\neq A)$. Let the line CB intersect Γ_2 in E ; let the line DB intersect Γ_1 in F. Prove that, the points C, D, E, F are concyclic.

---

4 Find all polynomials with real coefficients $P(x)$ such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

---

5 There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps.
(Assume that, the teacher has an abundant supply of apples.)

---

6 Let $\mathbb{N}$ denote set of all natural numbers and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a)$f(mn) = f(m).f(n)$ for all $m, n \in \mathbb{N}$;

(b)$m + n$ divides $f(m) + f(n)$ for all $m, n \in \mathbb{N}$.

Prove that, there exists an odd natural number k such that $f(n) = n^k$ for all n in $\mathbb{N}$.

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35th Indian National Mathematical Olympiad-2020

Time: 4 hours **January 19, 2020**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let Γ_1 and Γ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, in the plane intersecting in two distinct points A and B. Assume that the centre of each of the circles Γ_1 and Γ_2 is outside the other. The tangent to Γ_1 at B intersects Γ_2 again in C, different from B; the tangent to Γ_2 at B intersects Γ_1 again in D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet Γ_1 and Γ_2 again in X and Y, respectively, different from A. Let P and Q be the circumcentres of triangles ACD and XAY, respectively. Prove that PQ is the perpendicular bisector of the line segment O_1O_2.

2. Suppose $P(x)$ is a polynomial with real coefficients satisfying the condition

$$P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta),$$

for every real θ. Prove that $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(1-x^2)^2 + a_2(1-x^2)^4 + \cdots + a_n(1-x^2)^{2n},$$

for some real numbers $a_0, a_1, a_2, \ldots, a_n$ and nonnegative integer n.

3. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $S \subseteq X$ be such that any positive integer n can be written as $p + q$ where the non-negative integers p, q have all their digits in S. Find the smallest possible number of elements in S.

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be n real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 2n$. Prove that

$$a_1a_2\cdots a_{n-1} + a_1a_2\cdots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leq a_1a_2\cdots a_n.$$

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n \geq 3$ is called *frameable* if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.
   (a) Show that $3, 4, 6$ are *frameable*.
   (b) Show that any integer $n \geq 7$ is not frameable.
   (c) Determine whether 5 is *frameable*.

6. A *stromino* is a 3×1 rectangle. Show that a 5×5 board divided into twenty-five 1×1 squares cannot be covered by 16 *strominos* such that each *stromino* covers exactly three unit squares of the board and every unit square is covered by either one or two *strominos*. (A *stromino* can be placed either horizontally or vertically on the board.)

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India National Olympiad 2019

www.artofproblemsolving.com/community/c831794

by div5252

1 Let ABC be a triangle with $\angle BAC > 90$. Let D be a point on the segment BC and E be a point on line AD such that AB is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to AD. Given that $CA = CD$ and $AE = CE$. Determine $\angle BCA$ in degrees.

---

2 Let $A_1B_1C_1D_1E_1$ be a regular pentagon.For $2 \leq n \leq 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoint of the sides $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

---

3 Let m, n be distinct positive integers. Prove that

$$gcd(m, n) + gcd(m + 1, n + 1) + gcd(m + 2, n + 2) \leq 2|m - n| + 1.$$

Further, determine when equality holds.

---

4 Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3 \cdots, p_n$ such that p_j divides $M + j$ for all $1 \leq j \leq n$.

---

5 Let AB be the diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB.Let K be a point on the segment CD such that AC is equal to the semi perimeter of ADK.Show that the excircle of ADK opposite A is tangent to Γ.

---

6 Let f be a function defined from $((x, y) : x, y$ real, $xy \neq 0)$ to the set of all positive real numbers such that $(i) f(xy, z) = f(x, z) \cdot f(y, z)$ for all $x, y \neq 0$ $(ii) f(x, yz) = f(x, y) \cdot f(x, z)$ for all $x, y \neq 0$ $(iii) f(x, 1 - x) = 1$ for all $x \neq 0, 1$
Prove that $(a) f(x, x) = f(x, -x) = 1$ for all $x \neq 0$ $(b) f(x, y) \cdot f(y, x) = 1$ for all $x, y \neq 0$

The condition (ii) was left out in the paper leading to an incomplete problem during contest.

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India National Olympiad 2021

www.artofproblemsolving.com/community/c1962669

by anantmudgal09

1 Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2, \ldots, m_r, n_r$ be $2r$ integers such that

$$|m_i n_j - m_j n_i| = 1$$

for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

Proposed by B Sury

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2 Find all pairs of integers (a, b) so that each of the two cubic polynomials

$$x^3 + ax + b \text{ and } x^3 + bx + a$$

has all the roots to be integers.

Proposed by Prithwijit De and Sutanay Bhattacharya

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3 Betal marks 2021 points on the plane such that no three are collinear, and draws all possible segments joining these. He then chooses any 1011 of these segments, and marks their midpoints. Finally, he chooses a segment whose midpoint is not marked yet, and challenges Vikram to construct its midpoint using **only** a straightedge. Can Vikram always complete this challenge?

Note. A straightedge is an infinitely long ruler without markings, which can only be used to draw the line joining any two given distinct points.

Proposed by Prithwijit De and Sutanay Bhattacharya

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4 A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves, the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.

Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

Proposed by Anant Mudgal

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5 In a convex quadrilateral $ABCD$, $\angle ABD = 30^\circ$, $\angle BCA = 75^\circ$, $\angle ACD = 25^\circ$ and $CD = CB$. Extend CB to meet the circumcircle of triangle DAC at E. Prove that $CE = BD$.

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Proposed by BJ Venkatachala

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6 Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients. Find all functions $f : \mathbb{R}[x] \to \mathbb{R}[x]$ satisfying the following conditions:

- f maps the zero polynomial to itself,
- for any non-zero polynomial $P \in \mathbb{R}[x]$, deg $f(P) \leq 1 + \deg P$, and
- for any two polynomials $P, Q \in \mathbb{R}[x]$, the polynomials $P - f(Q)$ and $Q - f(P)$ have the same set of real roots.

Proposed by Anant Mudgal, Sutanay Bhattacharya, Pulkit Sinha

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India National Olympiad 2022

www.artofproblemsolving.com/community/c3006124

by Flying-Man, hellomath010118, N1RAV

1 Let D be an interior point on the side BC of an acute-angled triangle ABC. Let the circumcircle of triangle ADB intersect AC again at $E(\neq A)$ and the circumcircle of triangle ADC intersect AB again at $F(\neq A)$. Let AD, BE, and CF intersect the circumcircle of triangle ABC again at $D_1(\neq A)$, $E_1(\neq B)$ and $F_1(\neq C)$, respectively. Let I and I_1 be the incentres of triangles DEF and $D_1E_1F_1$, respectively. Prove that E, F, I, I_1 are concyclic.

---

2 Find all natural numbers n for which there is a permutation σ of $\{1, 2, \ldots, n\}$ that satisfies:

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} = 0$$

---

3 For a positive integer N, let $T(N)$ denote the number of arrangements of the integers $1, 2, \cdots N$ into a sequence $a_1, a_2, \cdots a_N$ such that $a_i > a_{2i}$ for all i, $1 \leq i < 2i \leq N$ and $a_i > a_{2i+1}$ for all i, $1 \leq i < 2i + 1 \leq N$. For example, $T(3)$ is 2, since the possible arrangements are 321 and 312

(a) Find $T(7)$
(b) If K is the largest non-negative integer so that 2^K divides $T(2^n - 1)$, show that $K = 2^n - n - 1$.
(c) Find the largest non-negative integer K so that 2^K divides $T(2^n + 1)$

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India National Olympiad 2023

www.artofproblemsolving.com/community/c3259889

by anantmudgal09

1 Let S be a finite set of positive integers. Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ so that the product xy is a perfect square. Prove that one can find at least four distinct elements in S so that none of their pairwise products is a perfect square.

Note: As an example, if $S = \{1, 2, 4\}$, there are exactly five such ordered pairs: $(1, 1)$, $(1, 4)$, $(2, 2)$, $(4, 1)$, and $(4, 4)$.

Proposed by Sutanay Bhattacharya

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2 Suppose $a_0, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \ldots, 100\}$:

$$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k,$$

where indices are taken modulo 101, *i.e.*, $a_{100+i} = a_{i-1}$ for any i in $\{1, 2, \ldots, 100\}$. Show that it is impossible that each of these 101 polynomials has all its roots real.

Proposed by Prithwijit De

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3 Let $\mathbb{N}$ denote the set of all positive integers. Find all real numbers c for which there exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying:

- for any $x, a \in \mathbb{N}$, the quantity $\frac{f(x+a)-f(x)}{a}$ is an integer if and only if $a = 1$;
- for all $x \in \mathbb{N}$, we have $|f(x) - cx| < 2023$.

Proposed by Sutanay Bhattacharya

---

4 Let $k \geq 1$ and $N > 1$ be two integers. On a circle are placed $2N + 1$ coins all showing heads. Calvin and Hobbes play the following game. Calvin starts and on his move can turn any coin from heads to tails. Hobbes on his move can turn at most one coin that is next to the coin that Calvin turned just now from tails to heads. Calvin wins if at any moment there are k coins showing tails after Hobbes has made his move. Determine all values of k for which Calvin wins the game.

Proposed by Tejaswi Navilarekallu

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5 Euler marks n different points in the Euclidean plane. For each pair of marked points, Gauss writes down the number $\lfloor \log_2 d \rfloor$ where d is the distance between the two points. Prove that Gauss writes down less than $2n$ distinct values.

Note: For any $d > 0$, $\lfloor \log_2 d \rfloor$ is the unique integer k such that $2^k \leq d < 2^{k+1}$.

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Proposed by Pranjal Srivastava

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6 Euclid has a tool called *cyclos* which allows him to do the following:

- Given three non-collinear marked points, draw the circle passing through them.
- Given two marked points, draw the circle with them as endpoints of a diameter.
- Mark any intersection points of two drawn circles or mark a new point on a drawn circle.

Show that given two marked points, Euclid can draw a circle centered at one of them and passing through the other, using only the cyclos.

Proposed by Rohan Goyal, Anant Mudgal, and Daniel Hu

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